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A remark on Fermat's last theorem.

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H.J.A.Duparc en A.van Wijngaarden.

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A REMARK ON FERMAT'S LAST THEOREM

BY

H. J. A. DUPARC and A. VAN WIJNGAARDEN
(Mathematical Centre, Amsterdam)

1. In a recent paper by R. OBLATH [1]) lower bounds for z^p, satisfying

$$x^p + y^p = z^p \quad (x, y, z \text{ positive integers}; p > 2, \text{ prime}) \tag{1.1}$$

are given. As usual two cases are distinguished, viz.

Case I: $xyz \not\equiv 0 \pmod{p}$;

Case II: $xyz \equiv 0 \pmod{p}$.

In either case certain congruences are combined with numerical lower bounds of p to the following results

Case I: $z^p > 10^{4.5\times 10^9}$;

Case II: $z^p > 10^{3.2\times 10^6}$.

In this note it is shown that in case I by using the same lower bound $p \geqq 253747889$ of D. H. and EMMA LEHMER [2]) the following sharper result can be derived:

Case I: $z > 10^{6\times 10^9}$; $z^p > 10^{1.5\times 10^{18}}$.

2. In the following sections p denotes a prime > 7.

For sake of symmetry in (1.1) put $X = x$, $Y = y$, $Z = -z$, hence

$$X^p + Y^p + Z^p = 0. \tag{2.1}$$

With the restriction of case I ($p \nmid xyz$) one has:

$$X,\ Y,\ Z \text{ are integers},\ p \nmid XYZ.$$

LANDAU [3]) proves

$$2X = -A^p+B^p+C^p,\ 2Y = A^p-B^p+C^p,\ 2Z = A^p+B^p-C^p,$$

where A, B and C are integers and

$$X + A \equiv Y + B \equiv Z + C \equiv 0 \pmod{p^2};$$
$$X^{p-1} \equiv Y^{p-1} \equiv Z^{p-1} \equiv 1 \pmod{p^3}.$$

Hence

$$A+B+C \equiv -(X+Y+Z) \equiv -(X^p+Y^p+Z^p) = 0 \pmod{p^2}.$$

Further [4])

$$-2CC' = A^p + B^p - C^p;\ (C, C') = 1;\ C' \equiv 1 \pmod{p^2}.$$

There are two kinds of prime factors of C, viz.

i) $q_1 \mid C$, $q_1 \nmid A + B$. From $q_1 \mid A^p + B^p$ a simple argument learns $q_1 \equiv 1 \pmod{p}$, hence $q_1^p \equiv 1 \pmod{p^2}$. Moreover using the first theorem of FURTWANGLER [5]) the prime factor q_1 of C, hence of Z satisfies

$$q_1 \equiv q_1^p \equiv 1 \pmod{p^2}.$$

ii) $q_2 \mid C$, $q_2 \mid A + B$. If $q_2^u \mid C$, $q_2^{u+1} \nmid C$, then $q_2^u \mid A^p + B^p$. Since $\left(A + B, \dfrac{A^p + B^p}{A + B}\right) = (A + B,\ p\, A^{p-1}) = 1$ (for otherwise either $p \mid AB \mid XY$ or $(A, B) \neq 1$) one has $q_2^u \mid A + B$, hence $q_2^u \mid A + B + C$.

Then putting $C = C_1C_2$, where C_1 only contains prime factors of the first kind (q_1) and C_2 only prime factors of the second kind (q_2) one has

$$C_1 \equiv 1 \pmod{p^2},\ C \equiv C_2 \pmod{p^2},\ C_2 \mid A + B + C. \tag{2.2}$$

Similarly

$$A \equiv A_2 \pmod{p^2},\ B \equiv B_2 \pmod{p^2} \tag{2.3}$$

and

$$A_2 \mid A + B + C,\ B_2 \mid A + B + C. \tag{2.4}$$

Since A, B and C are pairwise coprime, so are A_2, B_2 and C_2 hence

$$A_2B_2C_2 \mid A + B + C. \tag{2.5}$$

From $z > x$, $z > y$ it follows $A < 0$, $B < 0$ and from $x + y > 0$ it follows $C > 0$. Assuming without loss of generality $x < y$ one has $B < A$. Hence defining positive integers a, b, c and integers a_2, b_2, c_2 by

$$a + A = b + B = c - C = a_2 + A_2 = b_2 + B_2 = c_2 - C_2 = 0$$

one has

$$2x = a^p - b^p + c^p,\ 2y = -a^p + b^p + c^p,\ 2z = a^p + b^p + c^p. \tag{2.6}$$

Since $(x + y)^p > x^p + y^p = z^p$ one has $x + y > z$, $c^p > a^p + b^p$. Hence $c > b > a > 0$. Further the following congruences hold

$$a + b - c \equiv a_2 + b_2 - c_2 \equiv 0 \pmod{p^2}$$

and

$$0 = x^p + y^p - z^p \equiv x + y - z = c^p - a^p - b^p \equiv c - a - b \pmod{6}.$$

Thus

$$a + b - c \equiv 0 \pmod{6p^2}. \tag{2.7}$$

Finally in virtue of (2.2) and (2.3) one obtains

$$a = a_2 + a_3p^2,\ b = b_2 + b_3p^2,\ c = c_2 + c_3p^2, \tag{2.8}$$

where a_3, b_3 and c_3 are integers and in virtue of (2.5) one has

$$a_2b_2c_2 \mid a + b - c. \tag{2.9}$$

3. Putting $\frac{a}{c} = \alpha$, $\frac{b}{c} = \beta$ from (1.1) and (2.6) one obtains

$$(-\alpha^p + \beta^p + 1)^p + (\alpha^p - \beta^p + 1)^p = (\alpha^p + \beta^p + 1)^p;$$

$$0 < \alpha < \beta < 1;\ \alpha^p + \beta^p < 1. \tag{3.1}$$

Using after a suggestion of C. G. LEKKERKERKER for $0 < u < v$ the relation

$$p(v - u)u^{p-1} < v^p - u^p < p(v - u)v^{p-1}$$

one has

$$2p\alpha^p(1 - \alpha^p + \beta^p)^{p-1} < (\alpha^p - \beta^p + 1)^p < 2p\alpha^p(\alpha^p + \beta^p + 1)^{p-1},$$

hence

$$\alpha^p < (\alpha^p - \beta^p + 1)^p < 2p\alpha^p(1 + 2\beta^p)^p,$$

thus

$$\alpha < \alpha^p - \beta^p + 1 < (1 + 2\beta^p)\sqrt[p]{2p}.$$

Consequently one finds the result

$$1 - \beta^p < \alpha(1 + 2\beta^p)\sqrt[p]{2p} \tag{3.2}$$

and

$$2(1 - \beta^p) > \alpha - \alpha^p + 1 - \beta^p > \alpha,$$

thus

$$1 - \beta^p > \tfrac{1}{2}\alpha. \tag{3.3}$$

Now from (3.2) it follows

$$\beta > 1 - \frac{\log 2pe}{p}. \tag{3.4}$$

In fact the supposition $\beta \leq 1 - \frac{\log 2pe}{p}$ leads to

$$\beta^p < \left(1 - \frac{\log 2pe}{p}\right)^p < \frac{1}{2pe},$$

thus

$$e^{-\frac{2}{pe}} < \frac{1}{1 + \frac{2}{pe}} < \frac{1 - \frac{1}{2pe}}{1 + \frac{1}{pe}} < \frac{1 - \beta^p}{1 + 2\beta^p} < a\sqrt[p]{2p} < \beta\sqrt[p]{2p} < e^{-\frac{1}{p}},$$

which is impossible since $e > 2$.

Since $p \geq 8$ one obtains from (3.4) the relation $\beta > \frac{1}{2}$.

Then using (3.2) one finds

$$\frac{1 - \beta^p}{1 - \beta} \geq 1 + \beta + \beta^2 + \beta^3 + \beta^4 >$$

$$> 1\tfrac{1}{2} + 3\beta^p > \sqrt{2}(1 + 2\beta^p) > \sqrt[p]{2p}(1 + 2\beta^p) > \frac{1 - \beta^p}{a},$$

hence

$$a + \beta > 1. \tag{3.5}$$

4. From (2.7) and (3.5) one has

$$a + b = c + mp^2, \text{ where } 6 \mid m,\ m > 0.$$

Now two cases are distinguished

i. $m \geq p$. Then

$$c > a = c - b + mp^2 > mp^2 \geq p^3. \tag{4.1}$$

ii. $6 \leq m < p$. Using (2.9) one has $a_2 b_2 c_2 \mid m$, hence

$$|a_2| \leq m < p,\ |b_2| < p,\ |c_2| < p.$$

Further $0 < b_2 + b_3 p^2$, hence $b_3 p^2 > -b_2 > -p$, thus $b_3 \geq 0$ and

$$c_2 - b_2 + p^2(c_3 - b_3) = c - b > 0,$$

hence

$$c_3 - b_3 > \frac{b_2 - c_2}{p^2} > \frac{-2}{p},\ c_3 \geq b_3 \geq 0.$$

The case $c_3 = b_3$ is excluded.

In fact suppose $c_3 = b_3$. Then $c_2 - b_2 = c - b > 0$, hence

$$\beta = \frac{b_2 + b_3 p^2}{c_2 + c_3 p^2} = 1 - \frac{c_2 - b_2}{c_2 + c_3 p^2},$$

thus using (3.3)

$$1 - \tfrac{1}{2}a > \beta^p > 1 - \frac{p(c_2 - b_2)}{c_2 + c_3p^2},$$

hence

$$a < \frac{2p(c_2 - b_2)}{c_2 + c_3p}, \quad a < 2(c_2 - b_2)p < 4p^2,$$

which contradicts

$$a = c - b + mp^2 > 6p^2.$$

Consequently $c_3 > b_3$. Using (3.4) one has

$$1 - \frac{\log 2pe}{p} < \beta = \frac{b_3}{c_3}\left(1 + \frac{b_2}{b_3p^2}\right)\left(1 + \frac{c_2}{c_3p^2}\right)^{-1}.$$

Thus

$$\frac{b_3}{c_3} > \left(1 - \frac{\log 2pe}{p}\right)\left(1 - \frac{|c_2|}{c_3p^2}\right)\left(1 - \frac{|b_2|}{b_3p^2}\right) >$$

$$> \left(1 - \frac{\log 2pe}{p}\right)\left(1 - \frac{1}{c_3p}\right)\left(1 - \frac{1}{b_3p}\right) > 1 - \frac{\log 2pe + \dfrac{1}{c_3} + \dfrac{1}{b_3}}{p}$$

$$> 1 - \frac{3 + \log 2p}{p}.$$

Since $c_3 \geqq b_3 + 1$ one has

$$c_3 > \frac{p}{3 + \log 2p},$$

hence

$$c = c_2 + c_3p^2 > \frac{p^3}{3 + \log 2p} - p.$$

Consequently comparing (4.1) and (4.2) in both cases i and ii the result (4.2) holds.

5. Using (2.6) and (4.2) one finds

$$z > \tfrac{1}{2}c^p, \quad c > \frac{p^3}{3 + \log 2p} - p.$$

From $p \geqq 253747889$ one finds

$z > 10^{6\times 10^9}$, $z^p > 10^{1.5\times 10^{18}}$.

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5) E. LANDAU, ibidem, 315, theorem 1038.

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